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Note

Sum coloring and interval graphs: a tight upper bound for the minimum number of colors

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Dedicated to Prof. Mario Lucertini, who taught me Operations Research

Abstract

The SUM COLORING problem consists of assigning a color $c(v_i) \in Z^+$ to each vertex $v_i \in V$ of a graph $G = (V,E)$ so that adjacent nodes have different colors and the sum of the $c(v_i)$'s over all vertices $v_i \in V$ is minimized. In this note we prove that the number of colors required to attain a minimum valued sum on arbitrary interval graphs does not exceed $\min\{n; 2\chi(G) - 1\}$. Examples from the papers [Discrete Math. 174 (1999) 125; Algorithmica 23 (1999) 109] show that the bound is tight.

Keywords: Coloring; Interval graphs; Upper bound

1. Introduction and problem definition

A *feasible node-coloring* is an assignment of a non-negative integer (color) $c(v_i) \in Z^+$ to each node $v_i \in V$ of a graph $G = (V,E)$ such that adjacent nodes receive different colors. The SUM COLORING problem consists of finding a feasible node-coloring of a graph $G=(V,E)$ such that the sum of the $c(v_i)$'s over all vertices is minimized. The minimum value of such a sum is known as *chromatic sum*. The problem of determining the chromatic sum of a graph was first introduced in [11].

Actually, the SUM COLORING problem is a special case of the more general OPTIMUM COST CHROMATIC PARTITION problem (OCCP, for short) (see, for example [10,20]): given a graph $G = (V,E)$, with $n = |V|$ nodes, and a non-decreasing sequence $(k_1, k_2, \ldots, k_n)$ of color costs, find a feasible node-coloring such that the sum

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of the $k_{c(v_i)}$'s over all vertices is minimized. OCCP reduces to SUM COLORING if $(k_1, k_2, \ldots, k_n) = (1, 2, \ldots, n)$.

The *strength* of a graph G is the minimum number of colors needed to achieve a minimum valued solution. By $s(G)$ and $s_{\mathrm{OCCP}}(G)$ we shall denote the strength of a graph in an instance of SUM COLORING and of OCCP, respectively (clearly, all the results proved for $s_{\mathrm{OCCP}}(G)$ also hold for $s(G)$). In this paper we shall prove an upper bound on the strength $s(G)$ of an arbitrary interval graph G.

It is immediate to see that $\chi(G) \leqslant s(G) \leqslant n$ and $\chi(G) \leqslant s_{\mathrm{OCCP}}(G) \leqslant n$, where n denotes the number of nodes in the graph. It is worth observing that $s_{\mathrm{OCCP}}(G)$ may depend on the given sequence of color costs: in [15] it is proved that given a graph G with chromatic number $\chi(G) = t$, there always exists a sequence of color costs such that $s_{\mathrm{OCCP}}(G) = t$.

The papers [2,5,7,9,10,15–18] all contain some results about the strength of a graph, for at least one of the above problems. We shall now summarize their results. Given a graph G, in what follows, $\Delta(G)$, $\chi(G)$, and $\omega(G)$ denote the maximum degree of a node, the chromatic number, and the clique number, respectively (i.e. the maximum number of edges incident on a node of G, the minimum number of colors in a feasible node-coloring of G, and the number of nodes in a maximum sized induced complete subgraph of G).

General graphs: Both problems are NP-hard [11,12,19]. In [2] it is proved that for arbitrary non-negative integers $k \geqslant 2$ and $t > 0$, there exists a graph G with chromatic number $\chi(G) = k$ and strength $s(G) \geqslant k + t$. For an arbitrary graph G, there exists a minimum cost node-coloring solution for OCCP, such that $c(v_i) \leqslant \deg(v_i) + 1$ [7,10]. In [15] it is proved that $s_{\mathrm{OCCP}}(G) \leqslant \Delta(G) + 1$ on an arbitrary connected graph G which is not complete and not an odd cycle [15]. This last result recalls the Brooks theorem and becomes a necessary and sufficient condition for the strength of a graph in the SUM COLORING problem: $s(G) = \Delta(G) + 1$ if and only if G is a complete graph or an odd cycle [5]. Let $D(G)$ denote the *degeneracy* of G, that is, the smallest integer d such that G reduces to the empty graph by the successive removal of vertices having degree at most d [1,13,14,21] (by definition, it follows that the vertices of any graph G can be colored with at most $D(G) + 1$ colors, that is $\chi(G) \leqslant D(G) + 1$).[1] A bound for $s(G)$ which holds for an arbitrary graph G is $s(G) \leqslant \lceil (1 + D(G) + \Delta(G))/2 \rceil$ [5], which is shown [5] to be tight, even for trees. Also, in [9], for each k, a tree T_k is constructed with $s(T_k) = k$ and $\Delta(T_k) = 2k - 2$. Finally, in [5] it is conjectured that $s(G) \leqslant \lceil (\chi(G) + \Delta(G))/2 \rceil$ on an arbitrary graph G. A result which shows that also the topology of the graph plays an important role in computing the strength, and not only some measure defined on the graph, is the following: given an arbitrary graph G with chromatic number $\chi(G)$ and strength $s_{\mathrm{OCCP}}(G) > \chi(G)$, there exists a supergraph G' of G, with $V(G') = V(G)$ and $E(G') \supseteq E(G)$, such that $s_{\mathrm{OCCP}}(G') = \chi(G)$ [15].

[1] It is here worth recalling two measures that relate to $D(G)$: the first one is $\mathrm{sw}(G)$, the *Szekeres–Wilf number* of a graph G [21], defined as the maximum, over all induced subgraphs H of G, of the minimum degree of the vertices of H, that is $\mathrm{sw}(G) = \max\{\delta(H),\ \text{for all } H \subseteq G\}$; the second one is $\mathrm{col}(G)$, the *coloring number of* G [1,5], defined as the smallest integer d such that for some linear ordering $v_1, v_2, \ldots, v_n$ of the vertices one has $|\{v_j\colon j < i,\ (v_i, v_j) \in E\}| < d$. It turns out that $\mathrm{sw}(G) = D(G)$, and $\mathrm{col}(G) = 1 + D(G)$.

All the above results hold for arbitrary graphs, and can be specialized for particular classes of graphs, for which the following bounds and theorems also hold.

Trees: OCCP can be solved in linear time on trees [10]. $s_{\mathrm{OCCP}}(T) \leqslant 1 + \lceil \Delta(T)/2 \rceil$ on an arbitrary tree T [15], thus also $s(T) \leqslant 1 + \lceil \Delta(T)/2 \rceil$ (note that, as $D(T) = 1$ on an arbitrary tree T, the bound for $s(G)$ can also be obtained [5,9] by specializing to trees the bound $s(G) \leqslant \lceil (1 + D(G) + \Delta(G))/2 \rceil$ which holds for arbitrary graphs, and which is tight even for trees [5]). Given an arbitrary tree T, one has $s_{\mathrm{OCCP}}(T) \leqslant 1 + \lfloor t/2 \rfloor$, where t denotes the number of vertices on a longest path [15]. This same bound is proved for $s(G)$ in [5], where it is stated that: given an arbitrary tree T, one has $s(T) \leqslant 1 + \lceil (\min\{d(T), \Delta(T)\}/2 \rceil$, where $d(T)$ denotes the number of edges in a longest path (diameter) of T. The two bounds coincide since $t = d(T) + 1$. The bound on $s_{\mathrm{OCCP}}(T)$ is tight, as proved by the infinite class of trees proposed in [15], together with suitable sequences of color costs, which allows for achieving the equality. Finally, it can be proved that given a sequence of color costs, there exist a tree T whose strength is $s_{\mathrm{OCCP}}(T) = n$ [15], and that given an arbitrary tree T with strength $s_{\mathrm{OCCP}}(G) \geqslant 2$, there exists a supertree $T' \supseteq T$ such that $s_{\mathrm{OCCP}}(T') = 2$ [15]. The strengths $s(\cdot)$ and $s_{\mathrm{OCCP}}(\cdot)$ on caterpillars are both not larger than 3 (caterpillars are those trees with 3 or more *leaves* (nodes with only one incident edge), which reduce to a path P after removal of all the leaves): in [3,17] exact linear algorithms are proposed which find optimum solutions to SUM COLORING and OCCP on caterpillars, respectively, in at most 3 colors. On paths, both SUM COLORING and OCCP are trivially solvable, and the strengths $s(\cdot)$ and $s_{\mathrm{OCCP}}(\cdot)$ verify $s(\cdot) = s_{\mathrm{OCCP}}(\cdot) = 2$.

Interval graphs: Interval graphs are the intersection graphs of intervals of a real line [4], and SUM COLORING (thus OCCP) is NP-complete on them [8,18,22]. However, OCCP can be solved in polynomial time on interval graphs if there are only two different values for the color costs, and it is NP-complete if the color costs may assume any of four different values [10]. Also, $s(G) = \chi(G)$ on an interval graph G which represent the intersections of: a set of intervals, none of which contains another one (these are exactly the *unit* (or *proper*) *interval graphs* [4]); a set of intervals any two of which either do not intersect, or one contains the other one; and a set of intervals each of which has length not larger than 3 (assuming that endpoints have only integer valued coordinates) [18]. In that paper it is conjectured that $s(G) \leqslant 2\chi(G) - 1$ on an arbitrary interval graph G, and in the present paper we are going to prove this conjecture.

Other classes of graphs: For the OCCP problem it is proved that given a sequence of color costs, there exist a planar block G whose strength is $s_{\mathrm{OCCP}}(G) = n$ [15]. In [16] it is proved that $s_{\mathrm{OCCP}}(\cdot)$ can be arbitrarily large on maximal outerplanar and maximal planar graphs, and a conjecture of Harary and Plantholt on $s_{\mathrm{OCCP}}(\cdot)$ for line graph is disproved. In [7], it is proved that an optimum solution to OCCP restricted to graphs with constant treewidth needs at most $\mathrm{O}(\log n)$ different colors (that is $s_{\mathrm{OCCP}}(G) = \mathrm{O}(\log n)$), and that the bound is tight: a tree and a graph with constant treewidth, with a sequence of color costs are proposed, such that any optimum solution for OCCP needs $\Omega(\log n)$ different colors. $s_{\mathrm{OCCP}}(G) = r$ on a complete r-partite graph G [15].

Related topics: The papers [2,6,12,23] discuss of the minimum number of vertices of a graph with a chromatic number $\chi(G)$ and strength $\chi(G) + t$, for any non-negative integer t.

In this paper we shall prove that $s(G) \leqslant \min\{n; 2\chi(G) - 1\}$ holds for an arbitrary interval graph G. The result on $s(G)$ proves the conjecture posed in [18]. The upper bound we propose is tight: in [18] a class of interval graphs is characterized, which requires exactly $2\chi(G) - 1$ colors to achieve the chromatic sum (it is worth noticing that the polynomial time ε-approximate algorithm ($\varepsilon < 2$) proposed in [18] for solving SUM COLORING on arbitrary interval graphs, makes use of at most $2\chi(G)-1$ colors). The exact algorithm proposed in [3] for SUM COLORING caterpillars makes use of three colors, showing that the bound is tight.

2. The strength of arbitrary interval graphs

In the present section, a *clique* is a subset of nodes, maximal under node-inclusion, which induces a complete subgraph, a *proper interval graph* is the intersection graph of a set of intervals none of which contains another one (also known as *unit interval graphs* [4]), and $V(H)$ denotes the node set of graph H. Recall that interval graphs are perfect graphs, hence $\chi(\cdot) = \omega(\cdot)$ holds on the interval graph itself and on any induced subgraph of it.

Theorem 1. *Let $G = (V, E)$ be an arbitrary interval graph. Then $s(G) \leqslant 2\chi(G) - 1$.*

Proof. By induction. Let $\mathscr{G}_i$ be the family of all the interval graphs G with chromatic number $\chi(G) = i$, and define $\sigma_i = \max\{s(G)\colon G \in \mathscr{G}_i\}$ as the maximum strength of an interval graph in the family $\mathscr{G}_i$. Consider $\mathscr{G}_1$, and let $G \in \mathscr{G}_1$. Since $\omega(G) = 1$, G is an independent set itself and SUM COLORING admits a unique optimal solution, which clearly makes use of 1 color. Thus $s(G) = 1$, and $\sigma_1 = 1$. Consider $\mathscr{G}_i$, with $i \geqslant 1$, and an arbitrary graph $G \in \mathscr{G}_i$. Partition the node set V of G into two subsets V' and $V'' = V - V'$, such that the subgraph G' induced by node set V' is a subgraph of G with the maximum number of nodes and chromatic number $\chi(G') \leqslant i - 1$. Indeed $|V''| \geqslant 1$. Since $|V'|$ is maximum, it is the case that $\chi(G')$ has value exactly $i - 1$, thus $G' \in \mathscr{G}_{i-1}$. Before going on with the proof of Theorem 1, we need to prove the following two lemmas, where G'' denotes the subgraph induced by node set V'', and we assume that a consecutive clique arrangement (c.c.a., for short) of the cliques of G is given (that is a linear order $C_1, C_2, C_3, \ldots,$ of the cliques of G such that, for every vertex x of G, the cliques which contain x occur consecutively; interval graphs are characterized by the existence of such an arrangement [4]).

Lemma 2. $\chi(G'') \leqslant 2$.

Proof. Assume, by contradiction, that $\chi(G'') \geqslant 3$, and consider three nodes x, y, z belonging to a maximum clique of G''. Define five collections $\mathscr{C}_3$, $\mathscr{C}_1^-$, $\mathscr{C}_2^-$, $\mathscr{C}_2^+$, $\mathscr{C}_1^+$ of cliques of G, as follows. $\mathscr{C}_3$ is the (never empty) family of the cliques of G containing all the three vertices x, y, z (note that, because of the c.c.a., the indices of the cliques of G belonging to $\mathscr{C}_3$ are consecutive, that is, a clique C_j of G belongs to $\mathscr{C}_3$ iff $m \leqslant j \leqslant M$, where m and M denote the minimum and maximum index of the cliques in $\mathscr{C}_3$). The remaining four (possibly empty) collections $\mathscr{C}_1^-$, $\mathscr{C}_2^-$, $\mathscr{C}_2^+$, $\mathscr{C}_1^+$ are, respectively: the family of the cliques of G whose index is smaller than m and which contains one vertex out of x, y, z, only; the family of the cliques of G whose index is smaller than m and which contains exactly two vertices out of x, y, z, the family of the cliques of G whose index is larger than M and which contain exactly two vertices out of x, y, z, and the family of the cliques of G whose index is larger than M and which contain one vertex out of x, y, z, only.

Consider the vertex, say y, out of x, y, z which does not belong to the vertex set of any clique in $\mathscr{C}_1^- \cup \mathscr{C}_1^+$ (it is easy to verify that such a vertex does always exist). We claim that the chromatic number $\chi(G(V' \cup \{y\}))$ of the graph $G(V' \cup \{y\})$ induced by vertex set $V' \cup \{y\}$ does not exceed $i-1$. In order to prove the claim, consider an arbitrary clique C_j of G. The following relations clearly hold:

$$|V(C_j) \cap (V' \cup \{y\})| = |V(C_j) \cap V'| = i-1$$

$$\text{for any } C_j \notin (\mathscr{C}_1^- \cup \mathscr{C}_2^- \cup \mathscr{C}_3 \cup \mathscr{C}_2^+ \cup \mathscr{C}_1^+),$$

$$|V(C_j) \cap (V' \cup \{y\})| = |V(C_j) \cap V'| = i-1 \text{ for any } C_j \in \mathscr{C}_1^- \cup \mathscr{C}_1^+,$$

$$|V(C_j) \cap (V' \cup \{y\})| \leqslant 1 + |V(C_j) \cap V'| \leqslant i-1 \text{ for any } C_j \in \mathscr{C}_2^- \cup \mathscr{C}_2^+,$$

and

$$|V(C_j) \cap (V' \cup \{y\})| \leqslant 1 + |V(C_j) \cap V'| \leqslant i-2 \text{ for any } C_j \in \mathscr{C}_3.$$

That is, $|V(C_j) \cap (V' \cup \{y\})| \leqslant i-1$ for any C_j clique of G. On the other hand, $\max\{|V(C_j) \cap (V' \cup \{y\})|$, for any C_j clique of $G\} = \omega(G(V' \cup \{y\})) = \chi(G(V' \cup \{y\}))$. Thus $\omega(G(V' \cup \{y\})) = \chi(G(V' \cup \{y\})) \leqslant i-1$, as claimed.

The just proved claim contradicts the hypothesis that the subgraph G' induced by node set V' is a subgraph of G with a maximum number of nodes and chromatic number $\chi(G') \leqslant i-1$, and the lemma follows. □

Lemma 3. *G'' is a proper interval graph.*

Proof. By the preceding lemma and the fact that indeed $|V''| \geqslant 1$, either $\chi(G'') = 1$ or $\chi(G'') = 2$. If $\chi(G'') = 1$, then G'' is an independent set, and the lemma is proved. If $\chi(G'') = \omega(G'') = 2$, let x, y be two adjacent vertices of G'' If, say, $y \in V(C_j)$ implies $x \in V(C_j)$ then, clearly $\omega(G(V' \cup \{y\})) = \chi(G(V' \cup \{y\})) \leqslant i-1$. In fact $|V(C_j) \cap V'| = |V(C_j) \setminus \{x, y\}| \leqslant i-2$ for any C_j clique of G s.t. $y \in V(C_j)$. This contradicts the hypothesis that $|V'|$ is maximum. Henceforth, there always exist a clique of G'' which contains x but not y, and a different one which contains y but not x, proving the claim. □

Proof of Theorem 1 (continued). Recalling that the strength of a proper interval graph equals its chromatic number [18], we get $s(G'') = \chi(G'') = \omega(G'') \leqslant 2$. Since $s(G') \leqslant \sigma_{i-1}$, we can write $s(G) \leqslant \sigma_{i-1} + 2$, and $\sigma_i \leqslant \sigma_{i-1} + 2$. From this we derive $\sigma_i \leqslant 2i-1$, as $\sigma_1 = 1$. Recalling that, by definition, $\chi(G) = i$ for all graphs $G \in \mathscr{G}_i$, the claimed thesis follows. □

Fig. 1. The “difficult” example $B^{[3]}$ [18], where $n^{[3]} = 50$, $\Delta(G^{[3]}) = 25$, $\chi(G^{[3]}) = \omega(G^{[3]}) = 3$, $s(G^{[3]}) = 2\chi(G^{[3]}) - 1 = 5$, and $D(G^{[3]}) = 2$.

By the above theorem and the general bound $s(G) \leqslant n$ which holds for arbitrary graphs G, we get that

Theorem 4. *Let G be an arbitrary interval graph. Then* $s(G) \leqslant \min\{n, 2\chi(G) - 1\}$.

The following examples show that the bounds are tight. When the interval graph G is exactly the complete graph K_n on n nodes, clearly, $s(G) = n$, verifying the bound $s(G) = \Delta(G) + 1$ by Hajiabolhassan et al. [5] for complete graphs. Caterpillars, which are the interval graphs with $\chi(G) = 2$, and the so-called “difficult” example discussed in [18], show that the bound $2\chi(G) - 1$ is tight (see [3,18]). In fact, the chromatic sum is achieved in at most 3 colors on caterpillars, and in at least $2\chi(\cdot) - 1$ on the difficult example. The difficult example $G^{[d]}$, defined for any positive integer d, is the intersection graph of a particular set $B^{[d]}$ of intervals, and has $n^{[d]} = 6^{d-1} + 2\sum_{k=0}^{d-2} 6^k$ nodes, maximum degree $\Delta(G^{[d]}) = n^{[d]}/2$, chromatic number $\chi(G^{[d]}) = \omega(G^{[d]}) = d$, strength $s(G^{[d]}) \geqslant 2\chi(G^{[d]}) - 1$, and degeneracy $D(G^{[d]}) = d - 1$ (in Fig. 1, the set $B^{[3]}$ is drawn). Note that all the bounds on $s(G)$ proposed in the literature are very loose on $B^{[d]}$, since their right-hand sides are function of $\Delta(G^{[d]})$, which has an exponential dependence on d. On the contrary, the upper bound $2\chi(G^{[d]}) - 1$ grows linearly in d and, in fact, is tight.

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